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ON GEOMETRY OF YOUNG DIAGRAMS FOR ARNOLD PERMUTATIONS

Abstract. In this paper we study geometry of Young diagrams for the special class of permutation called (C, B, A)-permutations (or Arnold permutations). In particular we compute the averages of such Young diagrams and compare them with the averages of arbitrary Young diagrams.

1. Introduction

In [1] a method of studying permutations, which is based on the geometry of Young diagrams is suggested.

Let us consider an arbitrary permutation. This permutation can be decomposed into independent cycles. Ordering cycles' lengths descending, we obtain sequence $\{a_1, a_2, \ldots, a_h\}$, where h is the number of cycles and a_i are the lengths of corresponding cycles. Construct the set or single squares, whose first row contains a_1 squares, the second one contains a_2 squares, and so on. This figure is said to be *Young diagram of permutation.*

It is natural to consider the following geometric characteristics of Young diagram:

- *height* h of Young diagram,
- *length* $l := a_1$ of Young diagram,
- *area* $n := a_1 + \ldots + a_h$ of Young diagram,
- *vertical and horizontal asymmetries* $\mu := h/l$ *and* $\eta := l/h$ of Young diagram,
- *density* $\lambda := n/(h \cdot l)$ of Young diagram.

In [1] computations of Young diagrams' *averages* $\widehat{h}$, $\widehat{l}$, $\widehat{\mu}$ and $\widehat{\lambda}$ for permutations with $n \leqslant 7$ are provided and conjectures about asymptotics

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of these averages as $n \to \infty$ are formulated:

$$\widehat{h}(n) \sim c_1 \ln n, \quad \widehat{l}(n) \sim c_2 n, \quad \widehat{\lambda}(n) \sim c_3 / \ln n, \quad \widehat{\mu}(n) \sim c_4 \ln n / n.$$

A.M. Vershik informed us that these propositions had been proved in [5, 6].

Also V.I. Arnold suggested studying a special class of the following permutations. Consider set $\{1, 2, \ldots, n\}$. Divide it into three non-empty blocks $\{A, B, C\}$ (which consist of sequent numbers) of a, b and c sizes correspondingly and move them in the following way: $\{C, B, A\}$. The resulting permutation is said to be (C, B, A)*-permutation* (or *Arnold permutation*) and is denoted by $\sigma(a, b, c)$.

The problem os studying (C, B, A)-permutations is the discrete analog of *interval exchange transformations problem* (see [2]).

The aim of this paper is studying the geometry of the Young diagrams of Arnold permutations. In particular, we compute the averages of the Young diagrams for (C, B, A)-permutation. It appears that some of them coincide with Goncharov, Vershik and Shmidt asymptotics.

2. Height of Young diagram for Arnold permutation

Recall that in [4] the ergodic criterion of Arnold permutation was obtained. In terms of the Young diagrams ergodicity of permutation means that its Young diagram has height 1. Also in [4] the proportion of ergodic Arnold permutations was computed.

2.1. **Generalization of ergodic criterion.** The following theorem generalizes this result.

Theorem 1. *The height of the Young diagram for Arnold permutation* $\sigma(a, b, c)$ *is equal to* $h = \mathrm{GCD}(a + b, b + c)$.

Proof. Recall that quantities $\sigma(i) - i$, where $i = 1, \ldots, n$, are called *steps of permutation* σ.

There are only three steps S_C, S_B and S_A in (C, B, A)-permutations: $S_C = \sigma(i) - i$, where $\sigma(i) \in C$, $S_B = \sigma(i) - i$, where $\sigma(i) \in B$ and $S_A = \sigma(i) - i$, where $\sigma(i) \in A$.

It is clear that

$$S_C = a + b, \quad S_B = a - c, \quad S_A = -b - c.$$

Besides, $S_B = S_A + S_C$.

Now let us prove Theorem 1.

1. If $\mathrm{GCD}(S_C, S_B, S_A) = h$, then the number of cycles in permutation $\sigma(a, b, c)$ is not less than h, because in one cycle there can be only numbers comparable with each other by modulo h. Hence numbers 1, 2, ..., h belong to different cycles.

2. Now we prove that the number of cycles is not *more* than h. It's enough to prove that each cycle of our permutations contains *all* numbers comparable with each other by modulo h.

Consider an arbitrary cycle of permutation $\sigma(a, b, c)$. If we complete it once, we obtain: $xS_A + yS_B + zS_C = 0$ (here x is the number of steps S_A in cycle, y is the number of steps S_B and z is the number of steps S_C).

Hence we get $(x + y)(b + c) = (y + z)(a + b)$.

As $\mathrm{GCD}(a + b, b + c) = h$, then

$$x + y \geqslant \frac{a + b}{h} \qquad \text{and} \qquad y + z \geqslant \frac{b + c}{h}.$$

Adding these inequalities, we obtain:

$$x + 2y + z \geqslant \frac{a + 2b + c}{h}.$$

Assume that all numbers in our cycle are comparable with number k by modulo h, where $1 \leqslant k \leqslant h$. It is easy to see that $y \geqslant [\frac{a+b-k}{h}] - [\frac{a-k}{h}]$. Therefore, $x + y + z \geqslant [\frac{a+b+c-k}{h}] + 1 = [\frac{n-k}{h}] + 1$.

As there are $[\frac{n-k}{h}] + 1$ numbers less than $n + 1$, which are comparable with k by modulo h, then our cycle contains *all* numbers comparable with k by modulo h.

Hence, there are h cycles in our permutation. □

Remark. It follows from the proof of Theorem 1, that the cycles of (C, B, A)-permutation contain all numbers less than $n+1$ and comparable with each other by modulo h. In particular the length of Young diagram equals $[\frac{n-1}{h}] + 1$.

2.2. Proportion of Arnold permutations with h cycles.

Corollary 1. *Proportion $\delta_n(h)$ of Arnold permutations with h cycles and length n asymptotically equals $\frac{1}{\zeta(2)} \cdot \frac{1}{h^2}$ as $n \to \infty$ and h fixed (here ζ is Riemann zeta-function).*

Proof. Consider Arnold permutation $\sigma(a, b, c)$. Let us define

$$x := \frac{b + c}{h} = \frac{n - a}{h} \quad \text{and} \quad y := \frac{a + b}{h} = \frac{n - c}{h}.$$

Then it follows from Theorem 1 that the set of Arnold permutations with h cycles corresponds to the set of points with mutually-prime coordinates in

$$\triangle_n = \{(x, y) \in \mathbb{Z}^2 : x < n/h,\ y < n/h,\ x + y > n/h\}.$$

So, we should compute the proportion of points with mutually-prime coordinates in $\triangle_n$ as $n \to \infty$. We resort to the following Theorem.

Theorem 2 (Arnold Theorem of uniform distribution; see [3])**.** *The set of integer points with mutually-prime coordinates is distributed uniformly on the plane (Fig. 1) with the density* $1/\zeta(2) = 6/\pi^2$.

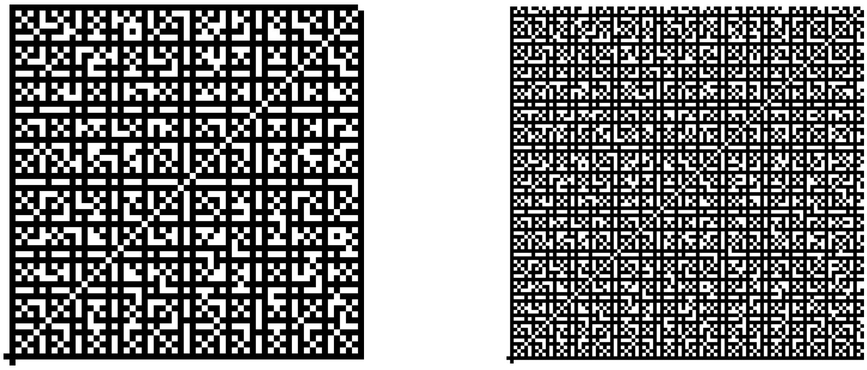

FIGURE 1. Uniform distribution: black points have mutually-prime coordinates, and white points don't.

So to prove our Theorem it's enough to apply the Arnold Theorem of uniform distribution to the convex bulls of sets $\triangle_n$. □

3. Young diagrams

Using corollary 1, one can find the averages of the Young diagrams for (C, B, A)-permutations. To do this we need the following definition.

3.1. Calendar Young diagrams.

Definition. Young diagram is said to be *calendar*, if the lengths of its rows differ not more than by 1.

Next corollary follows from Theorem 1.

Theorem 3. *All Young diagrams for* (C, B, A)*-permutations are calendar (Fig. 2, 3).*

Corollary 2. *Order of arbitrary Arnold permutation* $\leqslant \frac{n^2-1}{4}$.

Theorem 4. *Calendar Young diagram is uniquely defined by its height.*

Proof. Indeed, the lengths of the rows of calendar Young diagrams is uniquely expressed through its area and height. □

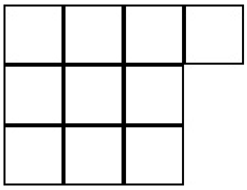

FIGURE 2. Permutation $\sigma(1,2,7)$

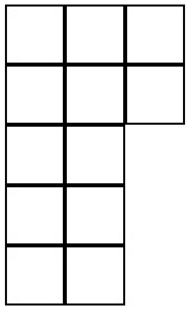

FIGURE 3. Permutation $\sigma(2,3,7)$

3.2. **Averages.** Theorem 4 and Corollary 1 make it possible to obtain the averages of the Young diagrams for Arnold permutations.

Theorem 5. *Asymptotics of the Young diagrams' average characteristics for Arnold permutations and arbitrary permutations are given in the following table. Here* $c \approx 0.62$ *is Golomb constant.*

Average	(C,B,A)***-permutations***	***Arbitrary permutations***
height $\widehat{h}$	$\frac{1}{\zeta(2)}\ln n$	$\ln n$
length $\widehat{l}$	$\frac{\zeta(3)}{\zeta(2)}n$	cn
density $\widehat{\lambda}$	1	$\frac{1}{\ln n}$
asymmetry $\widehat{\mu}$	$\frac{1}{\zeta(2)}$	$\frac{n}{\ln n}$
asymmetry $\widehat{\eta}$	$\frac{\zeta(4)}{\zeta(2)}n$	*?*

Table 1.

Proof. Let f be an arbitrary function on the Young diagrams for Arnold permutations. Then it follows from Theorem 4 that this function can be considered as function on height h of Young diagram. Hence its average $\widehat{f}$ equals

$$\widehat{f}(n) \sim \sum_{h=1}^{n} \delta_n(h) f(h) \sim \frac{1}{\zeta(2)} \cdot \sum_{h=1}^{n} \frac{f(h)}{h^2}.$$

For example, let us compute average length $\widehat{l}$ of Young diagram. Setting in the previous formula $f(h) = l(h) = \left[\frac{n-1}{h}\right] + 1$, we obtain

$$\widehat{l}(n) \sim \frac{1}{\zeta(2)} \sum_{h=1}^{n} \frac{1}{h^2}\left(\left[\frac{n-1}{h}\right] + 1\right) \sim \frac{1}{\zeta(2)} \sum_{h=1}^{n} \frac{n}{h^3} \sim \frac{\zeta(3)}{\zeta(2)} n.$$

Other asymptotics are proved in the same way. □

3.3. **Limit form.** Finally we study the *limit form* of the calendar Young diagrams.

Definition. Consider set of Young diagrams, whose first rows start with the common square. We rotate these diagrams with respect to this square by 90° and contract them by $\sqrt{n}$ times. Then this set of Young diagrams will fill in some figure as $n \to \infty$. This area is said to be *limit form of the set of Young diagrams.*

Theorem 6. *Limit form of the set of calendar Young diagrams is curvilinear triangle bound by the coordinate axes and by hyperbola $y = 1/x$ (Fig. 4, 5, 6).*

Proof. Consider an arbitrary calendar Young diagram. The length of its first row is equal to $[\frac{n-1}{h}] + 1$, the length of the next row is equal to $[\frac{n-2}{h}] + 1, \ldots,$ and the last row has length $[\frac{n-h}{h}] + 1$.

After the transformation of the Young diagram described above the columns of the diagram have height $([\frac{n-1}{h}] + 1)/n \sim 1/h$ as $n \to \infty$. Therefore the limit form is bound by coordinate axes and hyperbola $y = \frac{1}{x}$. □

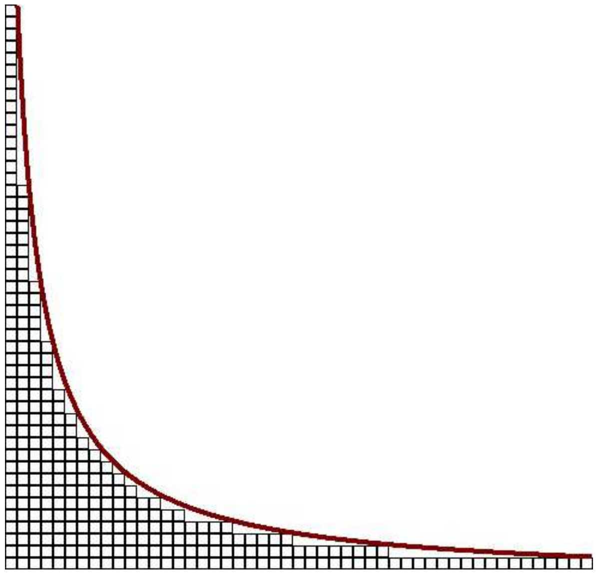

FIGURE 4. $n = 100$

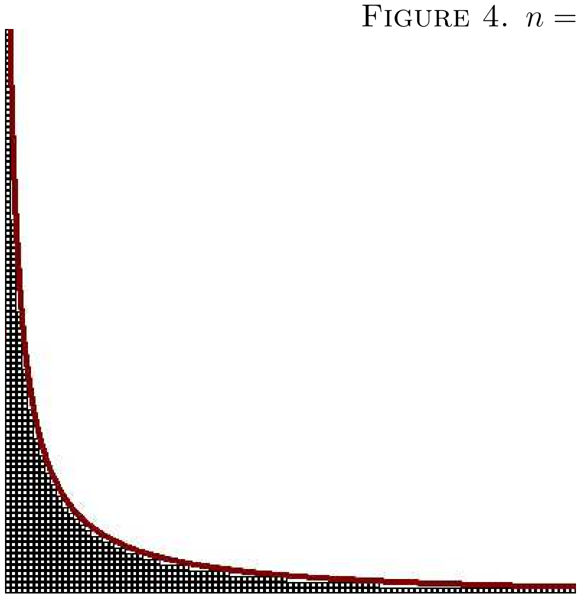

FIGURE 5. $n = 200$

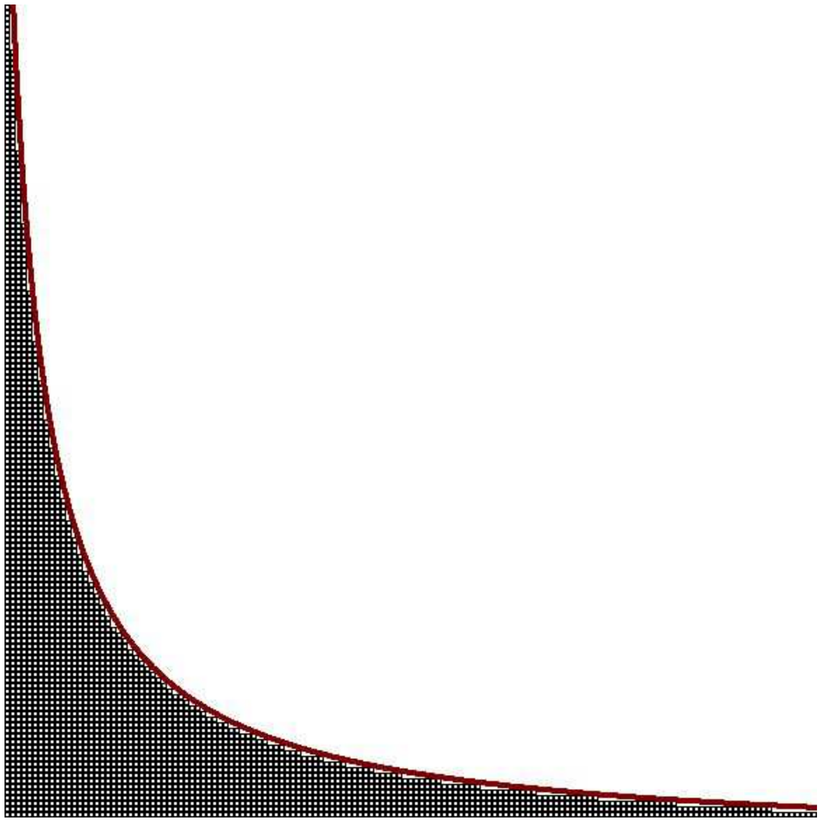

FIGURE 6. $n = 1000$

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